[1]

# A GEOMETRIC HAMILTON-JACOBI THEORY FOR CLASSICAL FIELD THEORIES

MANUEL DE LEÓN, JUAN CARLOS MARRERO,
AND DAVID MARTÍN DE DIEGO

Abstract. In this paper we extend the geometric formalism of the Hamilton-Jacobi theory for hamiltonian mechanics to the case of classical field theories in the framework of multisymplectic geometry and Ehresmann connections.

Contents



## 1. Introduction

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the **principal function**) such that

$$\frac{\partial S}{\partial t} + H(q^A, \frac{\partial S}{\partial q^A}) = 0. \tag{1.1}$$

[1]**To Prof. Demeter Krupka in his 65th birthday**

2000 *Mathematics Subject Classification.* 70S05, 49L99.

*Key words and phrases.* Multisymplectic field theory, Hamilton-Jacobi equations.

This work has been partially supported by MEC (Spain) Grants MTM 2006-03322, MTM 2007-62478, project "Ingenio Mathematica" (i-MATH) No. CSD 2006-00032 (Consolider-Ingenio 2010) and S-0505/ESP/0158 of the CAM.

If we put $S(t, q^A) = W(q^A) - tE$, where $E$ is a constant, then $W$ satisfies

$$H(q^A, \frac{\partial W}{\partial q^A}) = E; \tag{1.2}$$

$W$ is called the **characteristic function**.

Equations (1.1) and (1.2) are indistinctly referred as the **Hamilton-Jacobi equation**.

There are some recent attempts to extend this theory for classical field theories in the framework of the so-called multisymplectic formalism [15, 16]. For a classical field theory the hamiltonian is a function $H = H(x^\mu, y^i, p_i^\mu)$, where $(x^\mu)$ are coordinates in the space-time, $(y^i)$ represent the field coordinates, and $(p_i^\mu)$ are the conjugate momenta.

In this context, the Hamilton-Jacobi equation is [17]

$$\frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = 0 \tag{1.3}$$

where $S^\mu = S^\mu(x^\nu, y^j)$.

In this paper we introduce a geometric version for the Hamilton-Jacobi theory based in two facts: (1) the recent geometric description for Hamiltonian mechanics developed in [6] (see [8] for the case of nonholonomic mechanics); (2) the multisymplectic formalism for classical field theories [3, 4, 5, 7] in terms of Ehresmann connections [9, 10, 11, 12].

We shall also adopt the convention that a repeated index implies summation over the range of the index.

## 2. A geometric Hamilton-Jacobi theory for Hamiltonian mechanics

First of all, we give a geometric version of the standard Hamilton-Jacobi theory which will be useful in the sequel.

Let $Q$ be the configuration manifold, and $T^*Q$ its cotangent bundle equipped with the canonical symplectic form

$$\omega_Q = dq^A \wedge dp_A$$

where $(q^A)$ are coordinates in $Q$ and $(q^A, p_A)$ are the induced ones in $T^*Q$.

Let $H : T^*Q \longrightarrow \mathbb{R}$ a hamiltonian function and $X_H$ the corresponding hamiltonian vector field:

$$i_{X_H} \omega_Q = dH$$

The integral curves of $X_H$, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}, \; \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A}$$

**Theorem 2.1** (Hamilton-Jacobi Theorem)**.** *Let $\lambda$ be a closed 1-form on $Q$ (that is, $d\lambda = 0$ and, locally $\lambda = dW$). Then, the following conditions are equivalent:*

(i) *If $\sigma : I \to Q$ satisfies the equation*

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

*then $\lambda \circ \sigma$ is a solution of the Hamilton equations;*

(ii) $d(H \circ \lambda) = 0$.

To go further in this analysis, define a vector field on $Q$:

$$X_H^\lambda = T\pi_Q \circ X_H \circ \lambda$$

as we can see in the following diagram:

$$\begin{tikzcd}
T^*Q \arrow[r, "X_H"] \arrow[d, "\pi_Q"] & T(T^*Q) \arrow[d, "T\pi_Q"] \\
Q \arrow[r, "X_H^\lambda"] \arrow[u, bend left, "\lambda"] & TQ
\end{tikzcd}$$

Notice that the following conditions are equivalent:

(i) If $\sigma : I \to Q$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to Q$ is an integral curve of $X_H^\lambda$, then $\lambda \circ \sigma$ is an integral curve of $X_H$;

(i)" $X_H$ and $X_H^\lambda$ are $\lambda$-related, i.e.

$$T\lambda(X_H^\lambda) = X_H \circ \lambda$$

so that the above theorem can be stated as follows:

**Theorem 2.2** (Hamilton-Jacobi Theorem)**.** *Let $\lambda$ be a closed 1-form on $Q$. Then, the following conditions are equivalent:*

(i) *$X_H^\lambda$ and $X_H$ are $\lambda$-related;*

(ii) $d(H \circ \lambda) = 0$.

## 3. The multisymplectic formalism

3.1. **Multisymplectic bundles.** The configuration manifold in Mechanics is substituted by a fibred manifold

$$\pi : E \longrightarrow M$$

such that

(i) $\dim M = n,\ \dim E = n+m$
(ii) $M$ is endowed with a volume form $\eta$.

We can choose fibred coordinates $(x^\mu, y^i)$ such that

$$\eta = dx^1 \wedge \cdots \wedge dx^n \ .$$

We will use the following useful notations:

$$d^n x = dx^1 \wedge \cdots \wedge dx^n$$
$$d^{n-1}x^\mu = i_{\frac{\partial}{\partial x^\mu}}\, d^n x \ .$$

Denote by $V\pi = \ker T\pi$ the vertical bundle of $\pi$, that is, their elements are the tangent vectors to $E$ which are $\pi$-vertical.

Denote by

$$\Pi : \Lambda^n E \longrightarrow E$$

the vector bundle of $n$-forms on $E$.

The total space $\Lambda^n E$ is equipped with a canonical $n$-form $\Theta$:

$$\Theta(\alpha)(X_1, \dots, X_n) = \alpha(e)(T\Pi(X_1), \dots, T\Pi(X_n))$$

where $X_1, \dots, X_n \in T_\alpha(\Lambda^n E)$ and $\alpha$ is an $n$-form at $e \in E$.

The $(n+1)$-form

$$\Omega = -d\Theta \ ,$$

is called the canonical multisymplectic form on $\Lambda^n E$.

Denote by $\Lambda^n_r E$ the bundle of $r$-semibasic $n$-forms on $E$, say

$$\Lambda^n_r E = \{\alpha \in \Lambda^n E \mid i_{v_1 \wedge \cdots \wedge v_r}\alpha = 0, \text{ whenever } v_1, \dots, v_r \text{ are } \pi\text{-vertical}\}$$

Since $\Lambda^n_r E$ is a submanifold of $\Lambda^n E$ it is equipped with a multisymplectic form $\Omega_r$, which is just the restriction of $\Omega$.

Two bundles of semibasic forms play an special role: $\Lambda^n_1 E$ and $\Lambda^n_2 E$. The elements of these spaces have the following local expressions:

$$\begin{aligned} \Lambda^n_1 E &\ :\ p_0\, d^n x \\ \Lambda^n_2 E &\ :\ p_0\, d^n x + p^\mu_i\, dy^i \wedge d^{n-1}x^\mu \ . \end{aligned}$$

which permits to introduce local coordinates $(x^\mu, y^i, p_0)$ and $(x^\mu, y^i, p_0, p^\mu_i)$ in $\Lambda^n_1 E$ and $\Lambda^n_2 E$, respectively.

Since $\Lambda^n_1 E$ is a vector subbundle of $\Lambda^n_2 E$ over $E$, we can obtain the quotient vector space denoted by $J^1\pi^*$ which completes the following exact sequence of vector bundles:

$$0 \longrightarrow \Lambda^n_1 E \longrightarrow \Lambda^n_2 E \longrightarrow J^1\pi^* \longrightarrow 0 \ .$$

We denote by $\pi_{1,0} : J^1\pi^* \longrightarrow E$ and $\pi_1 : J^1\pi^* \longrightarrow M$ the induced fibrations.

3.2. **Ehresmann Connections in the fibration** $\pi_1 : J^1\pi^* \longrightarrow M$**.** A ***connection*** (in the sense of Ehresmann) in $\pi_1$ is a horizontal subbundle $\mathbf{H}$ which is complementary to $V\pi_1$; namely,

$$T(J^1\pi^*) = \mathbf{H} \oplus V\pi_1$$

where $V\pi_1 = \ker T\pi_1$ is the vertical bundle of $\pi_1$. Thus, we have:

(i) there exists a (unique) horizontal lift of every tangent vector to $M$;

(ii) in fibred coordinates $(x^\mu, y^i, p_i^\mu)$ on $J^1\pi^*$, then

$$V\pi_1 = \text{span}\,\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i^\mu}\}\ , \mathbf{H} = \text{span}\,\{\mathbf{H}_\mu\}\ ,$$

where $\mathbf{H}_\mu$ is the horizontal lift of $\frac{\partial}{\partial x^\mu}$.

(iii) there is a horizontal projector $\mathbf{h} : TJ^*\pi \longrightarrow \mathbf{H}$.

3.3. **Hamiltonian sections.** Consider a hamiltonian section

$$h : J^1\pi^* \longrightarrow \Lambda_2^n E$$

of the canonical projection $\mu : \Lambda_2^n E \longrightarrow J^1\pi^*$ which in local coordinates read as

$$h(x^\mu, y^i, p_i^\mu) = (x^\mu, y^i, -H(x,y,p), p_i^\mu)\ .$$

Denote by $\Omega_h = h^*\Omega_2$, where $\Omega_2$ is the multisymplectic form on $\Lambda_2^n E$.

The field equations can be written as follows:

$$i_{\mathbf{h}}\,\Omega_h = (n-1)\,\Omega_h\ , \tag{3.1}$$

where $\mathbf{h}$ denotes the horizontal projection of an Ehresmann connection in the fibred manifold $\pi_1 : J^1\pi^* \longrightarrow M$.

The local expressions of $\Omega_2$ and $\Omega_h$ are:

$$\begin{aligned}\Omega_2 &= -d(p_0\,d^n x + p_i^\mu\,dy^i \wedge d^{n-1}x^\mu)\\ \Omega_h &= -d(-H\,d^n x + p_i^\mu\,dy^i \wedge d^{n-1}x^\mu)\ .\end{aligned}$$

3.4. **The field equations.** Next, we go back to the Equation (3.1).

The horizontal subspaces are locally spanned by the local vector fields

$$H_\mu = \mathbf{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma_\mu^i\,\frac{\partial}{\partial y^i} + (\Gamma_\mu)_j^\nu\,\frac{\partial}{\partial p_j^\nu},$$

where $\Gamma_\mu^i$ and $(\Gamma_\mu)_j^\nu$ are the Christoffel components of the connection.

Assume that $\tau$ is an integral section of $\mathbf{h}$; this means that $\tau : M \longrightarrow J^1\pi^*$ is a local section of the canonical projection $\pi_1 : J^1\pi^* \longrightarrow M$ such that $T\tau(x)(T_xM) = \mathbf{H}_{\tau(x)}$, for all $x \in M$.

If $\tau(x^\mu) = (x^\mu, \tau^i(x), \tau_i^\mu(x))$ then the above conditions becomes

$$\frac{\partial \tau^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu}\ ,\ \frac{\partial \tau_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}$$

which are the Hamilton equations.

## 4. The Hamilton-Jacobi theory

Let $\lambda$ be a 2-semibasic $n$-form on $E$; in local coordinates we have

$$\lambda = \lambda_0(x,y)\, d^n x + \lambda_i^\mu(x,y)\, dy^i \wedge d^{n-1}x^\mu \; .$$

Alternatively, we can see it as a section $\lambda : E \longrightarrow \Lambda_2^n E$, and then we have

$$\lambda(x^\mu, y^i) = (x^\mu, y^i, \lambda_0(x,y), \lambda_i^\mu(x,y)) \; .$$

A direct computation shows that

$$d\lambda = \left( \frac{\partial \lambda_0}{\partial y^i} - \frac{\partial \lambda_i^\mu}{\partial x^\mu} \right) dy^i \wedge d^n x + \frac{\partial \lambda_i^\mu}{\partial y^j}\, dy^j \wedge dy^i \wedge d^{n-1}x^\mu \; .$$

Therefore, $d\lambda = 0$ if and only if

$$\frac{\partial \lambda_0}{\partial y^i} = \frac{\partial \lambda_i^\mu}{\partial x^\mu} \tag{4.1}$$

$$\frac{\partial \lambda_i^\mu}{\partial y^j} = \frac{\partial \lambda_j^\mu}{\partial y^i} \; . \tag{4.2}$$

Using $\lambda$ and $\mathbf{h}$ we construct an induced connection in the fibred manifold $\pi : E \longrightarrow M$ by defining its horizontal projector as follows:

$$\begin{aligned} \tilde{h}_e &: T_e E \longrightarrow T_e E \\ \tilde{h}_e(X) &= T\pi_{1,0} \circ h_{(\mu\circ\lambda)(e)} \circ \epsilon(X) \end{aligned}$$

where $\epsilon(X) \in T_{(\mu\circ\lambda)(e)}(J^1\pi^*)$ is an arbitrary tangent vector which projects onto $X$.

From the above definition we immediately proves that

(i) $\tilde{\mathbf{h}}$ is a well-defined connection in the fibration $\pi : E \longrightarrow M$.
(ii) The corresponding horizontal subspaces are locally spanned by

$$\tilde{H}_\mu = \tilde{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma_\mu^i((\mu\circ\lambda)(x,y))\, \frac{\partial}{\partial y^i} \; .$$

The following theorem is the main result of this paper.

**Theorem 4.1.** *Assume that $\lambda$ is a closed 2-semibasic form on $E$ and that $\tilde{h}$ is a flat connection on $\pi : E \longrightarrow M$. Then the following conditions are equivalent:*

(i) *If $\sigma$ is an integral section of $\tilde{h}$ then $\mu\circ\lambda\circ\sigma$ is a solution of the Hamilton equations.*
(ii) *The $n$-form $h\circ\mu\circ\lambda$ is closed.*

Before to begin with the proof, let us consider some preliminary results.

We have

$$(h \circ \mu \circ \lambda)(x^\mu, y^i) = (x^\mu, y^i, -H(x^\mu, y^i, \lambda_i^\mu(x,y)), \lambda_i^\mu(x,y)) \ ,$$

that is

$$h \circ \mu \circ \lambda = -H(x^\mu, y^i, \lambda_i^\mu(x,y))\, d^n x + \lambda_i^\mu \, dy^i \wedge d^{n-1}x^\mu \ .$$

Notice that $h \circ \mu \circ \lambda$ is again a 2-semibasic $n$-form on $E$.

A direct computation shows that

$$\begin{aligned} d(h \circ \mu \circ \lambda) &= -\left( \frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p_j^\nu} \frac{\partial \lambda_j^\nu}{\partial y^i} + \frac{\partial \lambda_i^\mu}{\partial x^\mu} \right) dy^i \wedge d^n x \\ &\qquad + \frac{\partial \lambda_i^\mu}{\partial y^j} \, dy^j \wedge dy^i \wedge d^{n-1}x^\mu \ . \end{aligned}$$

Therefore, we have the following result.

**Lemma 4.2.** *Assume $d\lambda = 0$; then*

$$d(h \circ \mu \circ \lambda) = 0$$

*if and only if*

$$\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p_j^\nu} \frac{\partial \lambda_j^\nu}{\partial y^i} + \frac{\partial \lambda_i^\mu}{\partial x^\mu} = 0 \ .$$

**Proof of the Theorem**

$(i) \Rightarrow (ii)$

It should be remarked the meaning of $(i)$.

Assume that

$$\sigma(x^\mu) = (x^\mu, \sigma^i(x))$$

is an integral section of $\tilde{\mathbf{h}}$; then

$$\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu} \ .$$

$(i)$ states that in the above conditions,

$$(\mu \circ \lambda \circ \sigma)(x^\mu) = (x^\mu, \sigma^i(x), \bar{\sigma}_j^\nu = \lambda_j^\nu(\sigma(x)))$$

is a solution of the Hamilton equations, that is,

$$\frac{\partial \bar{\sigma}_i^\mu}{\partial x^\mu} = \frac{\partial \lambda_i^\mu}{\partial x^\mu} + \frac{\partial \lambda_i^\mu}{\partial y^j} \frac{\partial \sigma^j}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}.$$

Assume $(i)$. Then

$$\begin{aligned}
&\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^\nu_j}\frac{\partial \lambda^\nu_j}{\partial y^i} + \frac{\partial \lambda^\mu_i}{\partial x^\mu} \\
&= \frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^\nu_j}\frac{\partial \lambda^\nu_i}{\partial y^j} + \frac{\partial \lambda^\mu_i}{\partial x^\mu}\,, \qquad \text{(since } d\lambda = 0\text{)} \\
&= \frac{\partial H}{\partial y^i} + \frac{\partial \sigma^j}{\partial x^\nu}\frac{\partial \lambda^\nu_i}{\partial y^j} + \frac{\partial \lambda^\mu_i}{\partial x^\mu}\,, \qquad \text{(since the first Hamilton equation)} \\
&= 0 \qquad \text{(since } (i)\text{)}
\end{aligned}$$

which implies $(ii)$ by Lemma 4.2.

$(ii) \Rightarrow (i)$

Assume that $d(h \circ \mu \circ \lambda) = 0$.

Since $\tilde{h}$ is a flat connection, we may consider an integral section $\sigma$ of $\tilde{h}$. Suppose that

$$\sigma(x^\mu) = (x^\mu, \sigma^i(x)).$$

Then, we have that

$$\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i}.$$

Thus,

$$\begin{aligned}
\frac{\partial \bar{\sigma}^\mu_j}{\partial x^\mu} &= \frac{\partial \lambda^\mu_j}{\partial x^\mu} + \frac{\partial \lambda^\mu_j}{\partial y^i}\frac{\partial \sigma^i}{\partial x^\mu}\,, \\
&= \frac{\partial \lambda^\mu_j}{\partial x^\mu} + \frac{\partial \lambda^\mu_i}{\partial y^j}\frac{\partial \sigma^i}{\partial x^\mu}\,, \qquad \text{(since } d\lambda = 0\text{)} \\
&= \frac{\partial \lambda^\mu_j}{\partial x^\mu} + \frac{\partial \lambda^\mu_i}{\partial y^j}\frac{\partial H}{\partial p^\mu_i}\,, \qquad \text{(since the first Hamilton equation)} \\
&= -\frac{\partial H}{\partial y^j}\,, \qquad \text{(since } (ii)\text{)}. \qquad \square
\end{aligned}$$

Assume that $\lambda = dS$, where $S$ is a 1-semibasic $(n-1)$-form, say

$$S = S^\mu \, d^{n-1}x^\mu$$

Therefore, we have

$$\lambda_0 = \frac{\partial S^\mu}{\partial x^\mu}\ ,\ \lambda^\mu_i = \frac{\partial S^\mu}{\partial y^i}$$

and the Hamilton-Jacobi equation has the form

$$\frac{\partial}{\partial y^i}\left(\frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i})\right) = 0\ .$$

The above equations mean that

$$\frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = f(x^\mu)$$

so that if we put $\tilde{H} = H - f$ we deduce the standard form of the Hamilton-Jacobi equation (since $H$ and $\tilde{H}$ give the same Hamilton equations):

$$\frac{\partial S^\mu}{\partial x^\mu} + \tilde{H}(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = 0 \ .$$

An alternative geometric approach of the Hamilton-Jacobi theory for Classical Field Theories in a multisymplectic setting was discussed in [15, 16].

## 5. Time-dependent mechanics

A hamiltonian time-dependent mechanical system corresponds to a classical field theory when the base is $M = \mathbb{R}$.

We have the following identification $\Lambda^1_2 E = T^*E$ and we have local coordinates $(t, y^i, p_0, p_i)$ and $(t, y^i, p_i)$ on $T^*E$ and $J^1\pi^*$, respectively. The hamiltonian section is given by

$$h(t, y^i, p_i) = (t, y^i, -H(t, y, p), p_i) \ ,$$

and therefore we obtain

$$\Omega_h = dH \wedge dt - dp_i \wedge dy^i \ .$$

If we denote by $\eta = dt$ the different pull-backs of $dt$ to the fibred manifolds over $M$, we have the following result.

The pair $(\Omega_h, dt)$ is a cosymplectic structure on $E$, that is, $\Omega_h$ and $dt$ are closed forms and $dt \wedge \Omega_h^n = dt \wedge \Omega_h \wedge \cdots \wedge \Omega_h$ is a volume form, where $dimE = 2n + 1$. The Reeb vector field $\mathcal{R}_h$ of the structure $(\Omega_h, dt)$ satisfies

$$i_{\mathcal{R}_h} \Omega_h = 0 \ , \ i_{\mathcal{R}_h} dt = 1.$$

The integral curves of $\mathcal{R}_h$ are just the solutions of the Hamilton equations for $H$.

The relation with the multisymplectic approach is the following:

$$\mathbf{h} = \mathcal{R}_h \otimes dt \ ,$$

or, equivalently,

$$\mathbf{h}(\frac{\partial}{\partial t}) = \mathcal{R}_h \ .$$

A closed 1-form $\lambda$ on $E$ is locally represented by

$$\lambda = \lambda_0 \, dt + \lambda_i \, dy^i.$$

Using $\lambda$ we obtain a vector field on $E$:

$$(\mathcal{R}_h)_\lambda = T\pi_{1,0} \circ \mathcal{R}_h \circ \mu \circ \lambda$$

such that the induced connection is

$$\tilde{\mathbf{h}} = (\mathcal{R}_h)_\lambda \otimes dt$$

Therefore, we have the following result.

**Theorem 5.1.** *The following conditions are equivalent:*

(i) $(\mathcal{R}_h)_\lambda$ *and* $\mathcal{R}_h$ *are* $(\mu \circ \lambda)$*-related.*
(ii) *The* 1*-form* $h \circ \mu \circ \lambda$ *is closed.*

**Remark 5.2.** An equivalent result to Theorem 5.1 was proved in [14] (see Corollary 5 in [14]). ⋄

Now, if

$$\lambda = dS = \frac{\partial S}{\partial t}\, dt + \frac{\partial S}{\partial y^i}\, dy^i ,$$

then we obtain the Hamilton-Jacobi equation

$$\frac{\partial}{\partial y^i}\left(\frac{\partial S}{\partial t} + H(t, y^i, \frac{\partial S}{\partial y^i})\right) = 0 .$$

## References


[1] R. Abraham, J.E. Marsden: *Foundations of Mechanics* (2nd edition). Benjamin-Cumming, Reading, 1978.

[2] V.I. Arnold: *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, Springer-Verlag, Berlin, 1978.

[3] E. Binz, J. Sniatycki, H. Fischer: *Geometry of classical fields*, North-Holland Mathematics Studies, **154**, North-Holland Publishing Co., Amsterdam, 1988.

[4] F. Cantrijn, A. Ibort, M. de León: On the geometry of multisymplectic manifolds. *J. Austral. Math. Soc. (Series A)* **66** (1999), 303–330.

[5] F. Cantrijn, A. Ibort, M. de León: Hamiltonian structures on multisymplectic manifolds. *Rend. Sem. Mat. Univ. Pol. Torino*, **54**, 3 (1996), 225–236.

[6] J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda, N. Román-Roy: Geometric Hamilton-Jacobi theory. *Int. J. Geom. Meth. Mod. Phys.* **3** (7) (2006), 1417-1458.

[7] M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery: *Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory.* arXiv:physics/9801019v2.

[8] M. de León, D. Iglesias-Ponte, D. Martín de Diego: Hamilton-Jacobi Theory for Nonholonomic Mechanical Systems. To appear in *J. Phys. A: Math. Theor.* 2007.

[9] M. de León, J.C. Marrero, D. Marín: A geometrical approach to Classical Field Theories: a constraint algorithm for singular theories. In: *New Developments in Differential Geometry (Debrecen, 1994), Mat. Appl.* **350**, Kluwer, Dordrecht, 1996, 291-312.

[10] M. de León, J.C. Marrero, D. Marín: Ehresmann connections in Classical Field Theories. In: *Differential Geometry and its Applications (Granada, 1994), Anales de Física, Monografías*, **2**, 1995, 73-89.

[11] M. de León, D. Martín de Diego, A. Santamaría-Merino: Tulczyjew' s triples and lagrangian submanifolds in classical field theory. In: *Applied Differential Geometry and Mechanics*, W. Sarlet and F. Cantrijn (eds). Universiteit Gent, 2003. pp. 21-47.

[12] M. de León, D. Martín de Diego, A. Santamaría-Merino: Symmetries in Field Theory. *International Journal of Geometric Methods in Modern Physics* **1** (5) (2004), 651-710.

[13] M. de León, P.R. Rodrigues: *Generalized Classical Mechanics and Field Theory*. North-Holland, Amsterdam, 1985.

[14] J.C. Marrero, D. Sosa: The Hamilton-Jacobi equation on Lie Affgebroids. *International Journal of Geometric Methods in Modern Physics* **3** (3), (2006), 605-622.

[15] C. Paufler, H. Romer: De Donder-Weyl equations and multisymplectic geometry. XXXIII Symposium on Mathematical Physics (Torún, 2001). *Rep. Math. Phys.* **49** (2002), 325-334.

[16] C. Paufler, H. Romer: Geometry of Hamiltonean $n$-vector fields in multisymplectic field theory. *J. Geom. Phys.* **44** (2002), 52-69.

[17] H. Rund: *The Hamilton-Jacobi theory in the Calculus of Variations*. Robert E. Krieger Publ. Co., Nuntington, N.Y. 1973.

[18] D.J. Saunders: *The geometry of jet bundles*, London Mathematical Society Lecture Notes Ser. 142, Cambridge Univ. Press, Cambridge, 1989.



Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Serrano 123, 28006 Madrid, Spain

*E-mail address*: `mdeleon@imaff.cfmac.csic.es`

J.C. Marrero: Departamento de Matemática Fundamental, Universidad de La Laguna, La Laguna, Canary Islands, Spain

*E-mail address*: `jcmarrer@ull.es`

D. Martín de Diego: Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Serrano 123, 28006 Madrid, Spain

*E-mail address*: `d.martin@imaff.cfmac.csic.es`